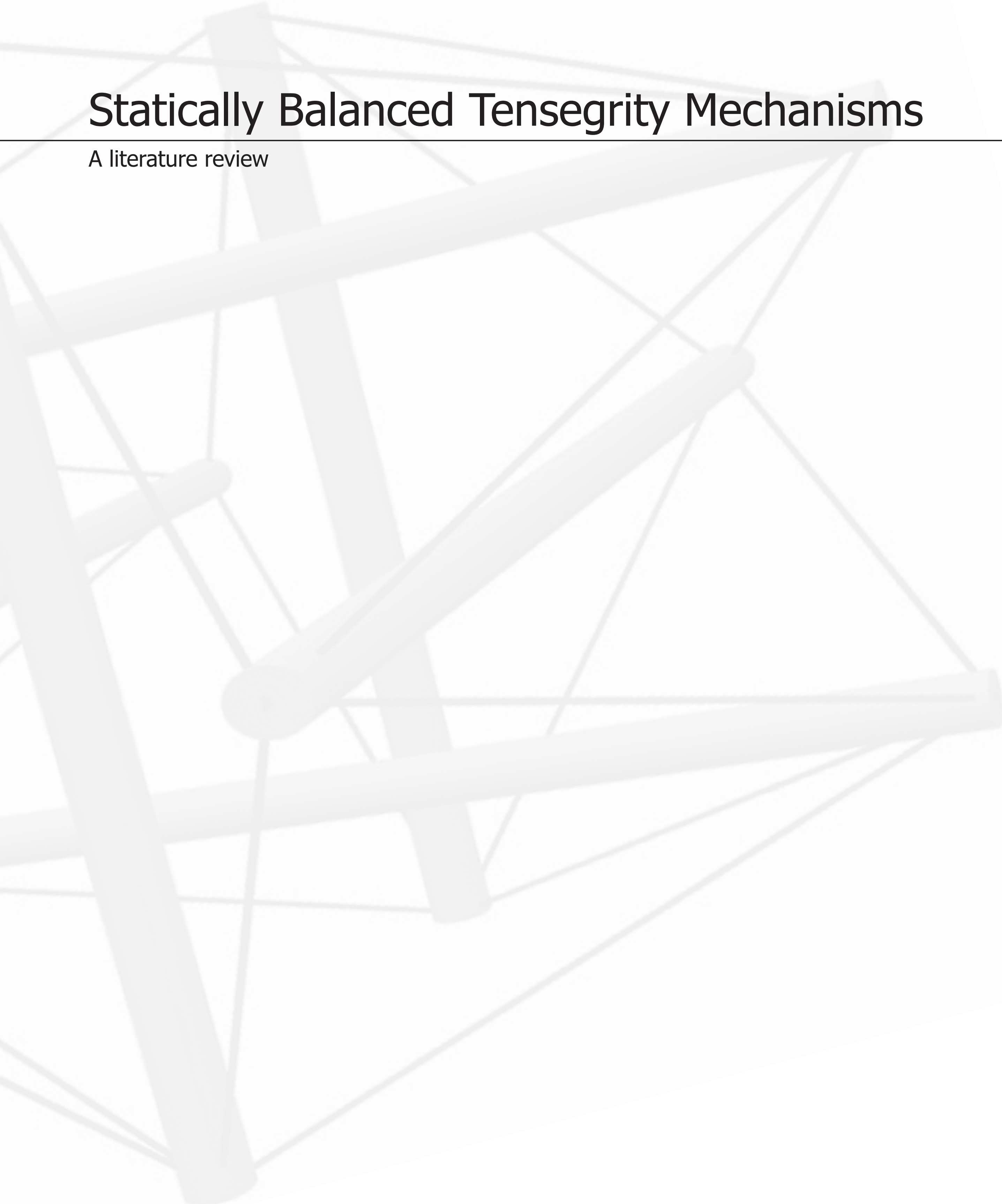
Statically Balanced Tensegrity Mechanisms

A literature review



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Preface

This report concludes the literature survey, which is a preliminary part of my M.Sc. research project at Delft University of Technology.

During this final project of my Mechanical Engineering study I will be working on the *theory and design of statically balanced tensegrity mechanisms*. The literature survey intends to establish the state of the art on the subject, and to provide a solid theoretical foundation for the actual M.Sc. thesis.

The bulk of this report was written during my internship at the Cambridge University Engineering Department, where I worked in the Structures Group. For being given the opportunity to work there for 4 months, and for all the valuable help and guidance during my stay, I would like to take this opportunity to thank Dr. Simon Guest.

Contents

Summary

Research in the field of *static balancing* has yielded some basic statically balanced *tensegrity structures*. This result holds a promise for combining the two concepts into more complex structures with potential engineering applications.

A review of statically balanced mechanisms showed that the governing principle of static balancing is *zero stiffness* or *neutral stability*. This provides a clear link towards the analysis of the structural stability of pin-jointed bar frameworks, of which tensegrities are a subset.

The first stage in tensegrity design and analysis is *form finding*, aimed at finding a self-stressed equilibrium geometry. Existing form-finding methods do not provide a great deal of flexibility, although the *force-density* approach can be augmented to include constraints and symmetry considerations. *Affine transformations* provide a new insight into the design of tensegrities and may provide a link to static balancing.

The discussion of the second stage, the *structural stability* analysis, ranges from Maxwell's rule to geometrically non-linear FEA. The latter includes all aspects of structural stability: geometry and topology, element properties and internal forces. It was shown that for the analysis of tensegrity structures the *tangent stiffness matrix* of non-linear FEA is required.

A recent derivation of the tangent stiffness matrix, which introduced the concept of *modified axial stiffness*, provided otherwise not immediately intuitive insight into the incorporation of zero-free-length springs into structural analysis. For zero-free-length springs the modified axial stiffness returns zero, and the tangent stiffness matrix only consists of the *stress matrix*. Another important observation is that zero-stiffness modes found in the tangent stiffness matrix can indicate static-balancing due to the zero-free-length springs, but also the presence of higher-order or finite mechanisms. No known methods yet exist to distinguish between these two types.

For the third and final stage, the load analysis, a brief overview has suggested the use of steering equations or arc-length methods for solving the non-linear FEA equations, due to the possiblity of limit points, snapbacks and the zero-stiffness modes.

Keywords: static balancing, zero-free-length springs, zero stiffness, tensegrity structures, prestressed structures, structural stability, neutral stability

Chapter 1

Introduction

This literature survey discusses the theoretical foundation for the investigation of the theory and design of *statically balanced tensegrity mechanisms*, a hitherto unexplorered combination of two fields of research, *tensegrity structures* and *statically balanced systems*.

Tensegrity structures, or tensegrities, are special types of prestressed bar frameworks with unique properties. Their engineering application is not limited to their architectural appeal, and they have also found their way into deployable and controllable structures [22, 27].

Statically balanced systems are in equilibrium in every configuration in their workspace, even when no friction is present. As a consequence, these systems can be operated with much less effort as compared to the unbalanced situation. Hence, static balancing is used for energy-efficient design in for instance prosthetics and rehabilitation technology [12].

The combination of the two fields is expected to produce mechanical frameworks with very interesting properties, that could provide new insight into both fields.

1.1 Scope and aims

The aim is to provide a solid theoretical foundation for combining tensegrities and statically balanced systems into a new field, that of statically balanced tensegrity mechanisms. In order to do so, both topics will first be introduced individually, establishing the state of the art, and throughout the discussion the combination of the two fields will be kept in mind.

In summary, this report will aim to obtain an overview of the principles of static balancing, including the properties of zero-free-length springs. Applying this knowledge in tensegrity structures, in turn requires an overview of tensegrity design and analysis. A main aspect of this, will be to recapitulate and clarify the structural analysis of bar frameworks, in order to expand its use to the combination of these with zero-free-length springs.

These points should provide a solid theoretical foundation and some ideas for promising routes for continuing research.

1.2 Layout

The report will be structured as follows. Chapter 2 will introduce the concept of tensegrity structures, including a brief historical overview and a listing of the engineering applications.

This is followed by a review of the principles of static balancing in chapter 3; the characteristics of zero-free-length springs are discussed, and the analysis of a basic spring force balancer will lead up to several examples of statically balanced tensegrities.

Chapter 4 will briefly place the three main aspects of the design and analysis of tensegrities into context: form-finding, structural stability and load analysis.

The following three chapters will focus on those individual aspects of the design and analysis. Main emphasis will lie on the structural stability of pin-jointed bar frameworks, which is already well-established in structural engineering, but which offers potential for new insights. The combination with static balancing will constantly be kept in mind, and consequences of the use of zero-free-length springs and the presence of zero-stiffness modes will be pointed out.

Chapter 8 will conclude the report with a short listing of possible research routes, which could provide more insight into the theory and design of statically balanced tensegrity mechanisms.

Chapter 2

Tensegrities

This section will provide a brief introduction to tensegrities. It will include a historical overview, a description of the concept and its scope, as well as a listing of some engineering applications.

2.1 History

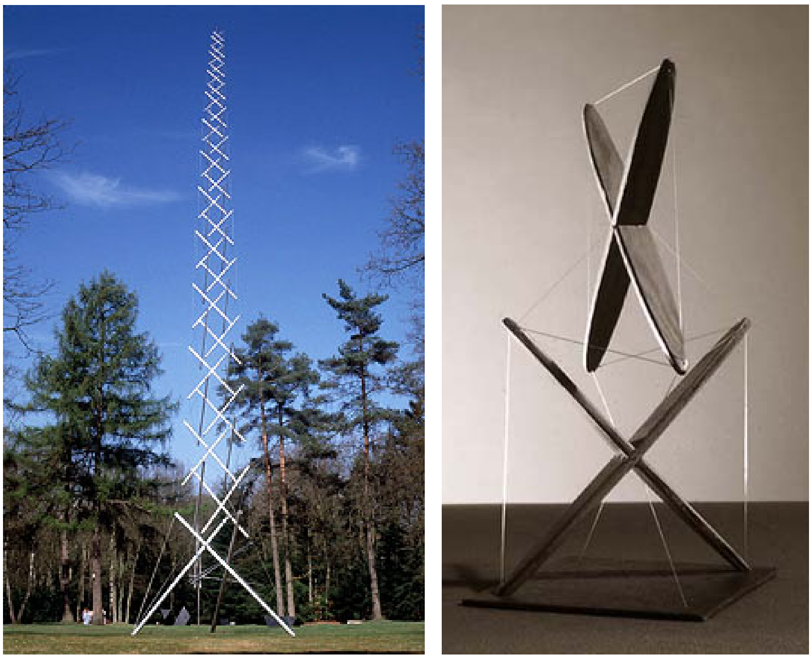
The origin of the word "*tensegrity*", which is a contraction of "*tens*ile int*egrity*", can be traced back to Buckminster Fuller [1] who coined the phrase in his 1962 patent application.

The origin of the tensegrity concept, however, is not as clearly established. The construction of the first true tensegrity structure is usually attributed to the artist Kenneth Snelson who created his *X-Piece* sculpture in 1948 (Figure 2.1), although some people point to the 1921 structure *Study in Balance* by Russian constructivist K. Ioganson as prior art.

The matter is clearly only of historical interest, as it are Buckminster Fuller and Snelson who contributed most to the conception of tensegrities. The former approached it from an engineering and a philosophical point of view, the latter from an artistic one, creating numerous tensegrity sculptures that convey the beauty and elegance of engineering.

Over the years many people have worked on tensegrities, although the topic has remained fairly obscure. Early studies were performed from a geometric point of view [e.g. 21], later followed by developments in structural analysis [notably 2, 20, 18]. Additional insights were provided by the use of mathematical Rigidity Theory [e.g. 4].

A more detailed historical overview of tensegrities is compiled by Motro [16] in a special issue of the *International Journal of Space Structures.*



(a) *Needle Tower II*, 1969 (b) *X-Piece*, 1948

Figure 2.1: Tensegrity sculptures by Kenneth Snelson

2.2 Description

The meaning and the scope of the word *tensegrity* is vague and many interpretations are possible. As a result, the concept goes by many names such as Snelson's *floating compression* and Emmerich's *selfstressed structures*.

It is not the intention of this section to provide the *The Definitive Definition* but rather to list several interesting definitions and descriptions which have surfaced over the years, in order to capture the essential aspects of tensegrities.

Definitions and descriptions

In his patent Snelson [23] describes tensegrities as a "*...class of structures possessing, what may be termed discontinuous compression, continuous tension characteristics.*" This discontinuity is also recognized by Buckminster Fuller [1] in his patent description, when he states that "*...the structure will have the aspect of continuous tension throughout and the compression will be subjugated so that the compression elements will become small islands in a sea of tension.*"

Pugh's elegant definition adds another important aspect, namely stability: "*A tensegrity system is established when a set of discontinuous compression components interact with a set of continous tensile components to define a stable volume in space.*"

A more mechanical description is given by for instance Hanaor [see 27] who describes tensegrity structures as "*... internally prestressed, free-standing pin-jointed networks, in which the cables or tendons are tensioned against a system of bars or struts.*" This description introduces the fact that the system is prestressed and pin-jointed; the latter addition is of great importance as it implies that there are no torques present in the system, only axial forces.

Another addition is provided by Pellegrino, who states that "*... as well as imparting tension to all cables, the state of prestress serves the purpose of stabilising the structure, thus providing first-order stiffness to its infinitesimal mechanisms.*"

Summary

Combined, the above descriptions cover most of the aspects of tensegrities, although maybe not all terms are equally meaningful at this stage. Let us now briefly summarize the key features of tensegrities:

pin-jointed bar frameworks Tensegrities belong to the structural group of pin-jointed bar frameworks; in other words, they form a special case of three-dimensional trusses.

pure compression/tension Tensegrity structures only contain pure compression and tension. What is more, tension elements are replaced by cables which can *only* sustain tension.

islands of compression in an ocean of tension In true tensegrity structures the compressive elements are disjunct; they seem to be floating among a continuous network of tension elements.

prestressed structures A state of prestress or self-stress is required for the stability of the structure; it stabilizes internal mechanisms.

2.3 Engineering applications

Let us commence this section by citing Snelson [24], who made an interesting remark concerning the practical application of tensegrity structures:

> *As I see it, this type of structure, at least in its purest form is not likely to prove highly efficient or utilitarian. As the engineer Mario Salvadori put it to me many years ago, "The moment you tell me that the compression members reside interior of the tension system, I can tell you I can build a better beam than you can." He was speaking metaphorically about this type of structure in general of course.*

Snelson makes an interesting point concerning the load handling of tensegrities, and thus their (in his view) limited practical relevance. It is also true that there have been few actual implementations of the tensegrity principle in engineering applications, which Motro [16] mainly ascribes to the lack of knowledge concerning actual construction methods.

Nevertheless, and aside from their architectural appeal, tensegrity structures are certainly relevant in various areas of engineering, due to some of their interesting traits.

self-stressed As tensegrities are self-stressed, it allows structures such as domes to be built without the need for supporting structures to equilibrate the stresses in the initial configuration.

light-weight For most materials, the tensile strength of a longitudinal member is larger than its buckling (compressive) strength. Hence, a large stiffness-to-mass ratio can be achieved by increasing the use of tensile members, as is the case with tensegrities.

deployable Another field of application is that of deployable structures in for instance aerospace engineering [27]. As the compressive members of tensegrity structures are disjoint, large displacements and thus deployability and compact stowage are possible.

energy-efficient Tensegrities are a subset of pin-jointed frameworks, but no pin-joints are actually required due to the bending flexibility of the cables. So if tensegrities were combined with the concept of static balancing, they could open the door to energy-efficient mechanisms where little to no energy is lost to friction in the joints.

variable stiffness Another interesting property can be found in the results of calculations by Skelton et al. [22]. When a tensegrity structure is subjected to a load, it provides a certain stiffness up to the point (which is determined by geometry and prestress conditions) where one of the strings slackens and the stiffness suddenly drops to a new value. This behaviour repeats itself for increasing loads, until the structure fails due to buckling or failure of the cables. This property could provide some degree of inherent safety when applied in the field of medical robotics, where contact forces need to be reduced to a minimum.

biotensegrity Some people argue that the tensegrity principle is a fundamental building block of life [13], and tensegrity-like behaviour has been observed in human cells. Others have noted the similarities between the bone/muscle structure of the body and the composition of tensegrities, and have used the tensegrity principle to model the human spine.

Drawbacks There are drawbacks to tensegrities as well, including their inferior rigidity and sensitivity to vibrations. Also, the failure of one of the components can be disastrous for the structure as a whole, or at least severely compromise the strength and structural integrity.

These problems have contributed to the limited number of practical applications of tensegrities, at least in their purest form.

Chapter 3

Static Balancing

In this section the concept of *static balancing* will be discussed and several key findings from Herder [12] [1] recapitulated, in order to arrive at some examples of statically balanced tensegrity structures.

3.1 Description

Efforts to define *statically balanced systems* have yielded three equivalent descriptions, each providing a different insight into the same concept:

- a statically balanced system is in static equilibrium throughout its range of motion, rather than in a single position or a limited number of positions only;
- the continuous equilibrium of the system implies a constant total potential energy throughout its motion;
- consequently, quasistatic motion requires no operating effort and the system has a *zero-stiffness*; in other words, the statically balanced system is in a state of neutral equilibrium or neutral stability, just in between stable and unstable.

In principle, any conservative force can be equilibrated. Obvious balancing methods include the use of counterweights or springs to balance a mass, but the most relevant balancing technique for this project is the *spring-to-spring balance* or *spring force compensation.*

[1] Unlike most research in static balancing, Herder [12] is not limited to a specific mechanism, but also discusses the governing principles of static balancing, which is of main interest to this project and should hence account for the lack of other literature sources used in this section.

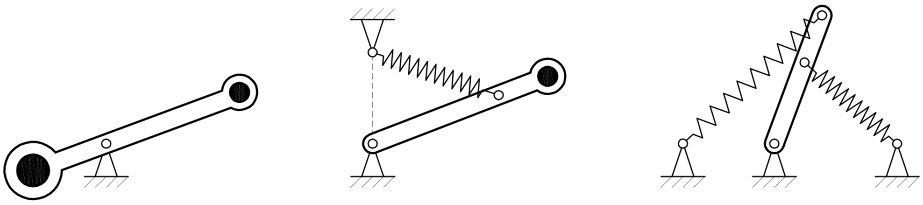


Figure 3.1: Various balancing methods; counterweight, spring-to-mass and spring-to-spring balancing.

3.2 Ideal springs

A common energy storage device is that of a helical extension spring. In the design of statically balanced systems a special type are favoured, namely *zero-free-length springs*, which tremendously simplify the conceptual design process and allow for perfect equilibrium.

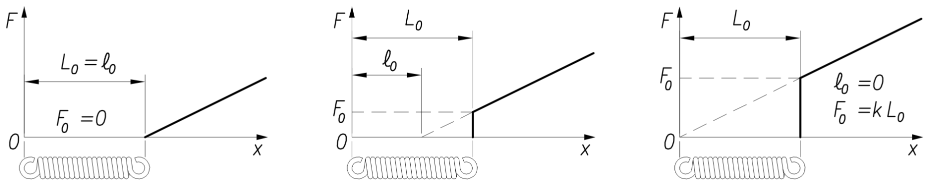


Figure 3.2: Spring characteristics of a normal spring, a normal spring with pretension and a zero-free-length spring.

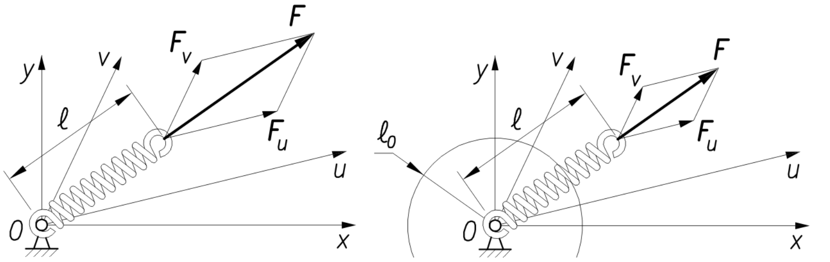
The free length of a spring should not be confused with the initial length. The initial length L_0 is the distance between the insides of the spring loops when no external load is present. The initial tension F_0 is the force needed to separate the coils at all, and the free length l_0 is defined as $L_0 - F_0/k$. So for zero-free-length springs, the exerted spring force is simply the length of the spring multiplied by the spring stiffness (see Figure 3.2 for details).

Additionally, the spring force in any directions u and v is simply equal to the excursion vectors in these directions multiplied by the spring stiffness, as can be seen in Figure 3.3. These simple relationships provide great computational advantage over normal springs.

Henceforth, extension springs with zero free length, constant spring stiffness, limitless strain, zero mass and forces acting along their centreline are termed ideal springs. Springs with a free length greater than zero will be called normal springs.

3.3 Basic spring force balancer

Using ideal springs, the *basic spring force balancer* takes the shape of Figure 3.4. This configuration can be derived and verified in a variety of ways, although the



(a) (b)

Figure 3.3: Resolution of spring forces in u and v direction: (a) ideal spring, where $\mathbf{F}_u = k\mathbf{u}$ and $\mathbf{F}_v = k\mathbf{v}$, (b) normal spring, where $\mathbf{F}_u = k(\mathbf{u} - l_0\mathbf{e}_u)$ and $\mathbf{F}_v = k(\mathbf{v} - l_0\mathbf{e}_v)$.

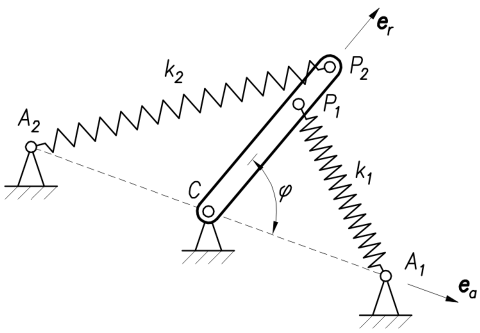


Figure 3.4: Basic spring force balancer, where $k_1 a_1 r_1 = k_2 a_2 r_2$ under condition that A_1, C, A_2 and C, P_1, P_2 are collinear.

various approaches all boil down to either the use of constant potential energy or zero-stability (which are equivalent). Here the conception will be taken for granted, but the balancing criteria will be verified by means of a potential energy function, first for one spring, then for the complete mechanism.

A configuration of a spring attached between ground and the end of a rotatable link, for example CP_1A_1 for spring k_1, will be called a *spring-lever element*. When the vectors $\mathbf{a}_{i/c} = CA_i$ and $\mathbf{r}_{i/c} = CP_i$ are defined, the potential of the spring-lever element i is

$$V_i = \frac{1}{2}k_i(\mathbf{a}_{i/c} - \mathbf{r}_{i/c})^T(\mathbf{a}_{i/c} - \mathbf{r}_{i/c}) \tag{3.1}$$

which, after the unit vectors $\mathbf{e}_r$ along the link and $\mathbf{e}_a$ running along C to A_1 are defined, can be rewritten as

$$\begin{aligned} V_i &= \frac{1}{2}k_i(a_i\mathbf{e}_a - r_i\mathbf{e}_r)^T(a_i\mathbf{e}_a - r_i\mathbf{e}_r) \\ &= \frac{1}{2}k_i(a_i^2 - 2a_ir_icos(\varphi) + r_i^2) \\ &= \frac{1}{2}k_i(a_i^2 + r_i^2) - k_ia_ir_icos(\varphi) \qquad (3.2) \end{aligned}$$

where φ is the angle between $\mathbf{e}_r$ and $\mathbf{e}_a$. Now, for the total system the potential energy $V = V_1 + V_2$ yields:

$$V = \frac{1}{2}k_i(a_1^2 + r_1^2) + \frac{1}{2}k_i(a_2^2 + r_2^2) - k_1 a_1 r_1 cos(\varphi) - k_2 a_2 r_2 cos(\pi - \varphi). \quad (3.3)$$

From this equation the moment equilibrium can be derived

$$V_{,\varphi} = k_1 a_1 r_1 sin(\varphi) - k_2 a_2 r_2 sin(\varphi) = 0 \quad (3.4)$$

and when requiring that this equality to zero holds for any φ and under the condition that both A_1,C,A_2 and C,P_1,P_2 are collinear, the following condition must be satisfied for static balancing:

$$k_1 a_1 r_1 = k_2 a_2 r_2. \quad (3.5)$$

The same result is obtained by demanding that the total potential energy is constant, which is the case when the cosine terms in equation 3.3 cancel.

3.4 Statically balanced tensegrity structures

This section will demonstrate the required steps to go from the basic static balancer to several statically balanced tensegrities. In the process, use will be made (both explicitly and implicitly) of the modification rules found in Herder [12, chap. 4], where their validity is also discussed.

The first step, as shown in Figure 3.5, involves the **kinematic inversion** of the whole mechanism. This clearly has no effect on relative motions of the elements and therefore leaves the system behaviour unchanged; it remains statically balanced.

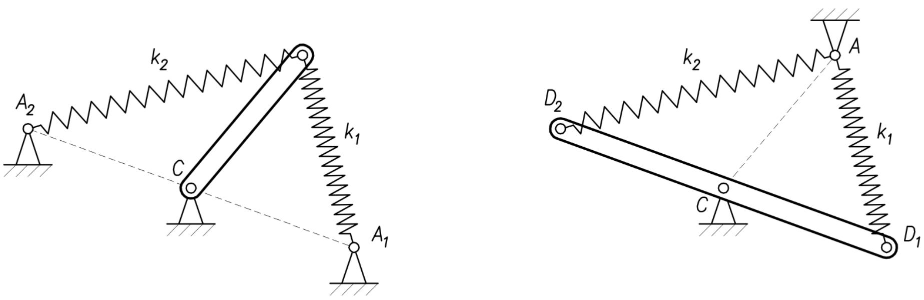


Figure 3.5: Inversion of springs and bar.

In addition to kinematic inversion, the interchange of springs and links is sometimes also possible. This interchange is only allowed when the links are purely

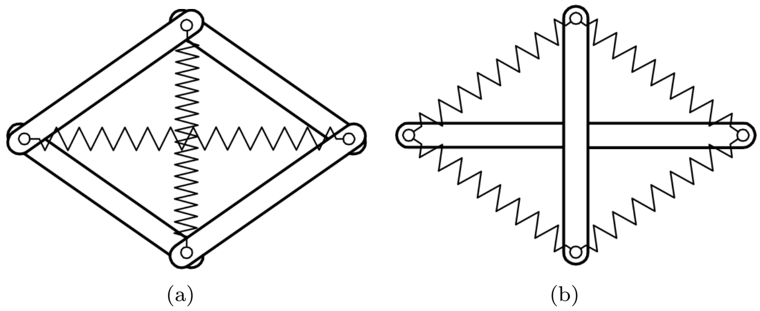


Figure 3.6: Basic shapes for statically balanced tensegrities.

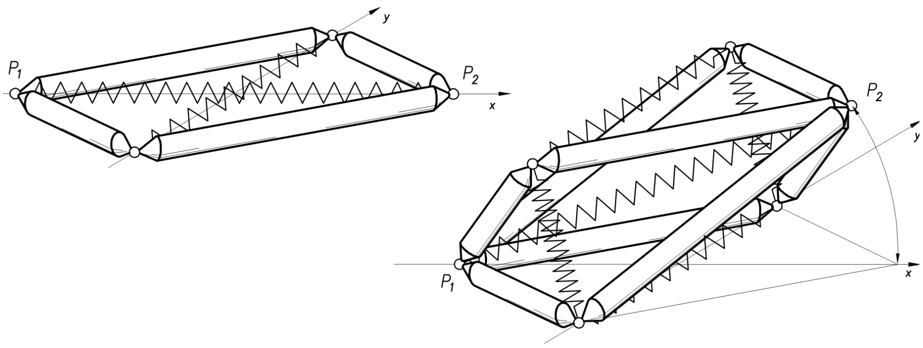


Figure 3.7: Conception of 3D Rhombus from 2D shape.

axially loaded, which is for example the case in the balanced rhombus (symmetric version of a parallelogram) of Figure 3.6(a). This inversion results in the statically balanced system of Figure 3.6(b).

The structures in Figure 3.6 can already be considered two-dimensional tensegrities (although they do not require pretension to maintain their shape). It is with these two basic shapes that we move on towards 3D tensegrities.

The simplest 3D version is created from Figure 3.6(b) by adding a link orthogonal to the existing two, intersecting the intersection of the original two links, and connecting the ends of all links with springs. This way a regular octahedron is formed, where any rotations of the links are statically balanced.

The creation of a slightly more complex statically balanced 3D tensegrity can be seen in Figure 3.7. It is created by overlaying two balanced rhombusses, connecting them at points P_1 and P_2, then expanding the structure and adding additional springs to connect the nodes.

3.5 Discussion

The examples shown in this section clearly show that there are existing cases of (3D) statically balanced tensegrity structures. There are almost certainly many more examples, which cannot be derived with the kind of rule-of-thumb methods applied previously.

Further developments in combining the fields of static balancing and tensegrity structures may require a different approach, which may yield a more generic way to design statically balanced structures. A good first step would be to search for zero-stiffness modes in structures, as it can be considered the fundamental governing principle behind static balancing.

Chapter 4

Mechanics of Tensegrities

This section aims to provide a backdrop for the coming chapters, by placing the three main stages of tensegrity design and analysis (form finding, structural stability and load analysis) into context.

4.1 Introduction

Due to their unique properties, tensegrities are rather complex structures to design and analyse. To give an idea of the matters involved in mechanics of tensegrities, let us start by quoting Motro [16], who wrote that the:

> *Definition of their geometrical shape [...] depends simultaneously on the initial geometry of the constituent elements (non-deformed lengths of cables and struts), the relational structure (topology) of the system and the existence of selfstressing, a necessary condition for a certain degree of rigidity.*

The structural stiffness of the assembly under loading is also a (highly) non-linear function of geometry, prestress and applied loads. In that scenario, the geometry plays the most significant role, which suggests that there is an optimum geometry with respect to for instance bending stiffness [22]. Pretension further serves the important role of maintaining stiffness until a cable goes slack.

4.2 Design approach

The design of tensegrities can be split up into three distinct stages. It should, however, be pointed out that in order to successfully design a tensegrity structure, these steps should not be performed consecutively, but should overlap, and intermediate results should be fed back to earlier stages.

Form finding

Finding a self-stressed equilibrium configuration is the first and arguably the most difficult step and is referred to as *form finding*. For other tension structures, several form-finding methods have successfully been employed, but tensegrities provide additional difficulties.

Structural Stability

Having found a self-stressed equilibrium configuration does not imply that the structure is actually stable. This question of stability is the topic of the next phase, *structural stability*, which includes the identification of internal mechanisms and self-stress states.

Chapter 6 will show that a lot of tools are available for this stage, and a lot of insight can be gained into the combination of pin-jointed bar frameworks with zero-free-length springs and the possibility of static balancing.

Load Analysis

Once existence of an acceptable solution has been shown, the behaviour of the tensegrity system can be studied under the influence of external actions. An important question is whether mechanisms can reappear as a result of external loading, or if the additional stresses further stabilize or destabilize the structure.

Chapter 5

Form finding

This section will provide an overview of the existing form-finding methods for tensegrities in order to determine a suitable method for designing statically balanced tensegrities.

5.1 Introduction

The first step in the analysis and design of tensegrity tructures is the determination of their equilibrium configuration, also known as form finding. During the form-finding process, tensegrity structures are usually considered under absence of external forces and constraints. For other tension structures, such as membrane and cable nets, efficient form-finding methods have been available for a long time. For general tensegrity structures, however, the form-finding process has proven to be more complicated.

In the past a geometrical approach making use of (semi-)regular polyhedra was used. However, numerical validation and physical models of the resulting shapes have shown that there is a deviation between the selfstressing form and the geometry of the polyhedra. This result emphasises the inadequacy of a purely geometrical approach [16].

Recently Tibert and Pellegrino [26] published a review and classification of the existing tensegrity form-finding methods, where they found 7 methods and classified them into kinematic and static methods. This section is largely based on their work.

5.2 Kinematic form-finding methods

The characteristics of the kinematic form-finding methods is that the lengths of the cables are kept constant while the bars are elongated until a *maximum* length is reached; or vice versa, where the bar lengths are fixed and the cables are shortened until they reach a *minimum*. This approach reflects how tensegrities are built in practice, by adjusting the lengths of the elements.

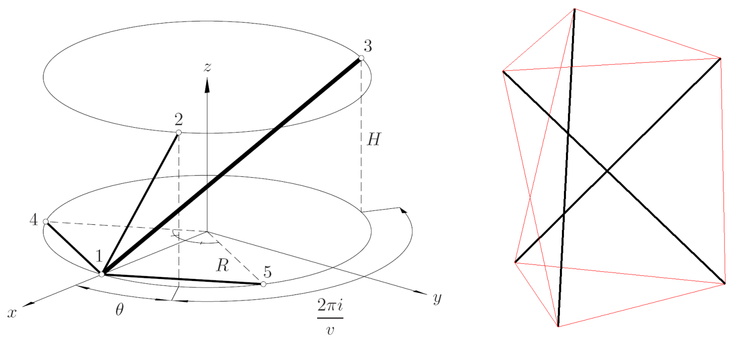


Figure 5.1: (a) Elements meeting at node 1 of a structure with v-fold symmetry, radius R and height H. (b) *Simplex* tensegrity with 3-fold symmetry.

5.2.1 Analytical solutions

Analytical solutions can be obtained only for very simple structures, such as tensegrity prisms where the equilibrium configuration is determined by relative rotation between the upper and lower regular polygon (see Figure 5.1). A derivation of the equilibrium conditions for a *Simplex* can for instance be found in Murakami [17].

5.2.2 Non-linear programming

The general method of non-linear programming works by turning form finding of any tensegrity structure into a constrained minimisation problem. Starting from a system with known element connectivity and nodal coordinates, one or more bars are elongated, maintaining fixed length rations, until a configuration is reached in which their lengths are maximized.

A general constrained minimisation problem in 3D has the form:

$$\begin{aligned} &\text{Minimise} && f(x,y,z) \\ &\text{subject to} && g_i(x,y,z) = 0 \quad \text{for} \quad i = 1,\dots,n. \end{aligned} \tag{5.1}$$

where the objective function $f(x,y,z)$ is for example the negative length of one of the bars and the constraint equations $g_i(x,y,z)$ are the fixed lengths of the cables.

An advantage of the non-linear programming approach is that it makes use of general purpose techniques, such as MATLAB's `fmincon` function. However, the number of constraint equations increases with the number of elements, so it is not feasible for large systems. Also, there is no direct way of controlling the variation in the state of selfstress, because the constraints are limited to element lengths.

5.2.3 Dynamic relaxation

The technique of dynamic relaxation turns a static problem into a fictious dynamic one. For a tensegrity structure in a given initial configuration and subject to given general forces, the equilibrium configuration can be computed by integrating the following fictitious dynamic equations

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{N}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{f}, \tag{5.2}$$

where the vector $\mathbf{d}$ contains displacements from the initial configuration, and both the mass matrix $\mathbf{M}$ and damping matrix $\mathbf{N}$ are assumed to be diagonal. The system is excited by increasing the length of the bars, calculating the resulting out-of-balance forces and corresponding accelerations.

Convergence is controlled by an appropriate choice of damping coefficients, or by a technique called kinetic damping. In the latter case the undamped motion of the structure is traced and when a local peak in the total kinetic energy is detected (i.e. minimum of potential energy), all velocity components are set to zero. The process is then repeated, starting from the current configuration, until the peak kinetic energy becomes sufficiently small.

Motro [see 26] concludes that the dynamic relaxation method has good convergence properties for structures with only a few nodes, but is not effective when the number of nodes increases. Also, the method becomes rather cumbersome if several different ratios between bar and cable lengths are desired.

5.3 Static form-finding methods

The general characteristic of these methods is that a relationship is set up between equilibrium configurations of a structure with a given topology and the forces in its members.

5.3.1 Analytical solutions

A general solution of the relative rotation between the upper and lower polygon of tensegrity prisms was derived by Connelly and Terrell [3].

5.3.2 Reduced coordinates

Consider a tensegrity structure of b elements, with M cables and O bars, where the bars can be seen as a set of bilateral constraints acting on the cable structure. Hence, a set of N independent, generalized coordinates $\mathbf{g} = (g_1 g_2 \ldots g_N)^T$ can be defined, which uniquely define the position and orientation of the bars.

For a state of self-stress of the structure, a set of cable forces $\mathbf{t} = (t_1 t_2 \ldots t_M)^T$ is in equilibrium with the appropriate forces in the bars and zero external loads. Now a set of equilibrium equations relating the forces in the cables, but without

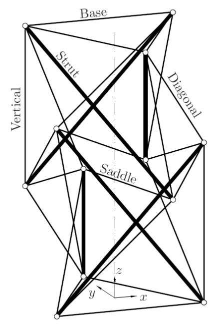


Figure 5.2: 2-stage tensegrity tower.

showing explicitly the forces in the bars, can be obtained from virtual work by assuming only changes in cable length during virtual displacements. Sultan et al. [25] used this method, which involves extensive symbolic computations, to find equilibrium configurations of a class of tensegrity towers (see Figure 5.2).

Using a set of well-chosen generalized coordinates and the principle of virtual work, Crane III et al. [6] also performed a static analysis of rotationally symmetric tensegrity structures.

5.3.3 Force density method

The linear force density method, successfully used for form finding of cable nets and membrane structures, employs a simple mathematical trick to transform the non-linear equilibrium equations at the nodes (e.g. node i in x-direction)

$$\sum_j \frac{t_{ij}}{l_{ij}}(x_i - x_j) = f_{ix} \tag{5.3}$$

into a set of linear ones, by introducing a force density

$$q_{ij} = \frac{t_{ij}}{l_{ij}} \tag{5.4}$$

with t_{ij} the tension in and l_{ij} the length of element ij. These force densities are chosen at the start of the form-finding process and are of great influence to the final outcome. The force density is also referred to as *tension coefficient* in literature. Note that it is identical to the spring stiffness of a zero-free-length spring, K_{zfl}.

For a general structure with b elements and n nodes, the equilibrium equations in the x-direction can be written as

$$\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{x} = \mathbf{f}_x \tag{5.5}$$

where $\mathbf{C}$ is the incidence matrix, $\mathbf{Q}$ a diagonal matrix containing the b force densities, $\mathbf{x}$ a column vector of x-coordinates, and $\mathbf{f}_x$ a column vector of external forces in x-direction; equivalent equations obviously hold for y and z-direction.

The incidence matrix $\mathbf{C}$ is of size $b \times n$ and describes the connectivity of the structure. It mostly contains zeros, except for entries belonging to member k with nodes i and j (where $i < j$), which are

$$c_{ki} = -1 \qquad \text{and} \qquad c_{kj} = 1. \tag{5.6}$$

When certain nodes are fixed to a foundation, equation 5.5 can be split into two parts. However, during form finding tensegrities are always modelled free-standing and under absence of external forces, so it reduces to

$$\mathbf{C}^T\mathbf{QCx} = \mathbf{0} \tag{5.7}$$

and with $\mathbf{D} = \mathbf{C}^T\mathbf{QC}$ it is rewritten as

$$\mathbf{Dx} = \mathbf{0}. \tag{5.8}$$

In structures consisting of cables only, all tension coefficients are positive, so $\mathbf{C}^T\mathbf{QC}$ is positive definite. Thus it is invertible and admits a unique solution. For tensegrities the $n \times n$ matrix $\mathbf{D}$ is semi-definite, due to the presence of compressive elements with $q_{ij} < 0$, and several complications arise during form finding.

To find a stable volume in three-dimensional space, the force density matrix $\mathbf{D}$ of the structure must have a nullity $\aleph = 4$ [30]. Finding a set of member force densities which fulfill this condition can be done via an intuitive, iterative or analytical method. The latter is the preferred approach, and requires the use of symbolic software.

Traditionally, the general setup of the force density method does not allow control over the lengths of the elements, making it suitable for finding new configurations, but limiting the practical application. However, Masic et al. [15] showed that certain shape constraints could be added to the form-finding process, for example to limit points to a plane or fix the length of an element.

More interestingly, Masic et al. showed how the number of variables in the form-finding process can be drastically reduced by considering symmetry of the structure. The paper also showed a more general theorem regarding affine transformations of tensegrity structures, and as it can be considered a form-finding method in its own right, it is discussed in section 5.3.5.

5.3.4 Energy method

The energy method is based on the mathematical Rigidity Theory and employed by for instance Connelly and Terrell [3]. It was found to be equivalent to the force density method, but with different terminology [26].

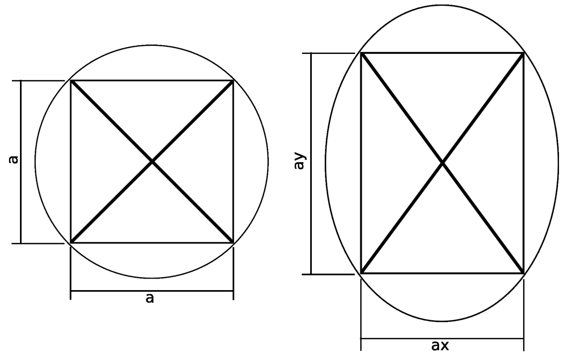


Figure 5.3: An equilibrium elliptical tensegrity cross generated by the similarity transform from the square configuration.

A mathematical energy potential for a tensegrity structure is set up, using force density and element length instead of elastic stiffness and element elongation. This potential has a local minimum for a structure in d-dimensional space if a square matrix $\mathbf{\Omega}$ is positive definite with a nullity of $d + 1$.

5.3.5 Affine transformation

Masic et al. [15] explored the effect of transformations, such as symmetry operations, on the form finding of tensegrities. Suppose now, that a tensegrity structure with nodes $\mathbf{p}$ is transformed using $\tilde{\mathbf{p}} = T(\mathbf{p})$ and if the new structure is also in equilibrium, then T is known as an *invariant tensegrity geometric transformation.*

The paper goes on to show that any *affine transformation* of the nodal position vector $\mathbf{p}$ is an invariant tensegrity geometric transformation. The affine geometric transformation is known as the *tensegrity similarity transformation.*

An affine transformation is any transformation that preserves collinearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation). In general, an affine transformation is a composition of rotations, translations, dilations, and shears [31].

Although it is intuitively obvious that a tensegrity structure remains in equilibrium if it undergoes a geometric transformation as a rigid body (i.e. translation or rotation) it is not obvious that the same holds for other affine transformations.

This opens a new approach to the design of tensegrities, because if one equilibrium position is found, any affine transformation will also be one, as is illustrated in Figure 5.3. It also provides a link to static balancing, because if an affine transformation can be found where the bars do not change length, but the zero-free-length springs do, that provides a possibly statically balanced mode.

5.4 Discussion

Tibert [27] concludes that in general the static methods seem to possess more usable features than the kinematic ones and that in search for new configurations, the force density method is well suited since the lengths of the elements are not specified at the start. The form-finding method can be further improved by adding constraints, for example to positions of points or element lengths, and by considering symmetry of the structure, which tremendously reduces the number of variables.

A recent and very promising insight is the observation that a tensegrity structure will remain in equilibrium under an affine transformation. So, if a certain stable geometry is found, it can be modified by means of affine transformations to create new structures. The affine transformation also provides insight into the static balancing of tensegrity structures, because if a zero-stiffness mode coincides with an affine transformation, it will likely be statically balanced.

The form-finding step will not be explored in this research. Focus will primarily lie on structural analysis, from which it is hoped that constraints and boundary conditions can be found for the form finding of statically balanced tensegrities, which might then be employed in novel form-finding methods.

Chapter 6

Structural stability

As was shown in chapter 3, the key to static balancing is the presence of a zero-stiffness mode, or equivalently, neutral stability. Therefore, the stability analysis of pin-jointed structures seems a logical next step in the search for statically balanced structures.

In this section the stability of pin-jointed bar frameworks will be discussed from the ground up, starting with Maxwell's rule which solely considers the number of elements and nodes in a structure. This is followed by linear structural analysis, which takes into account the geometry and topology of the structure. The final step is to consider the effect of material properties and prestresses as well, with non-linear FEA. It will be shown that the last step is necessary to obtain crucial insight into the effect of zero-free-length springs in pin-jointed structures.

To provide an additional point of view to the stability of structures, as a conclusion, the mathematical rigidity theory will be summarized, and parallels with standard structural analysis pointed out.

6.1 Maxwell's rule

In 1864 James Clerk Maxwell published an algebraic rule setting out a condition for a pin-jointed frame composed of b rigid bars and j joints to be both statically and kinematically determinate. The number of bars needed to stiffen a three-dimensional frame free to translate and rotate in space as a rigid body is given by

$$b = 3j - 6. \tag{6.1}$$

The physical reasoning behind the rule is clear: each added bar links two joints and removes at most one internal degree of freedom. The rule simply equates the number of external and internal degrees of freedom. For three dimensions, Maxwell's rule can generally be written as

$$b = 3j - c \tag{6.2}$$

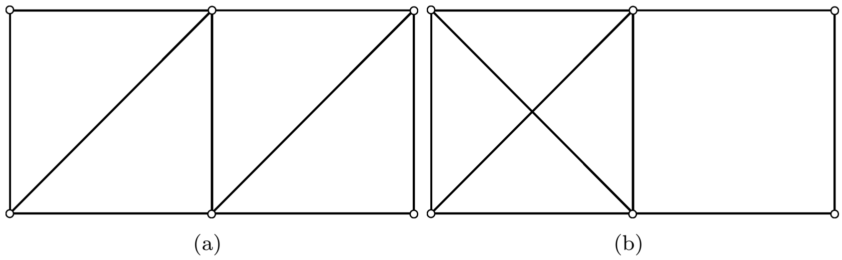


Figure 6.1: Two plane frames that satisfy Maxwell's rule, (a) is simply stiff, (b) is part redundant and part mechanism.

where c is the number of kinematic constraints ($c \geq 6$ in three dimensions). As Maxwell noted himself, equation 6.2 is a necessary, but not a sufficient condition for establishing determinacy. An obvious example can be seen in Figure 6.1 where both structures comply with Maxwell's rule, but the left one is obviously stiff, whereas the right one is partly redundant and partly a mechanism.

Calladine [2] extended Maxwell's original rule to include states of self-stress (bar tensions in the absence of external loads) and mechanisms (displacements of the joints without any bar extensions); these concepts will be clarified in the next sections. Now, the full inventory of degrees of freedom of the frame can be written as an extended Maxwell's rule:

$$3j - b - c = m - s \tag{6.3}$$

where s and m count the states of self-stress and mechanisms respectively, and can be found by finding the rank of the equilibrium matrix that describes the frame in a full structural analysis [20], and which will be elaborated on in this chapter.

The extended Maxwell rule can be illustrated by a simple example. Consider the two-dimensional framework in Figure 6.2 which consists of two bars, three joints and is subject to 4 kinematic constraints. The system complies with the extended rule with $s = 1$ and $m = 1$. Resolving the vertical equilibrium at the

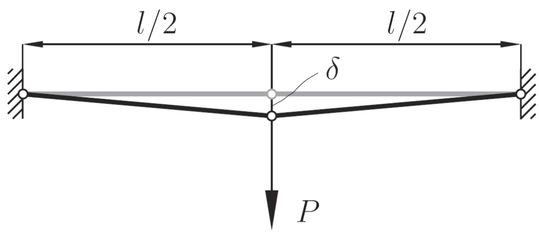


Figure 6.2: Two-bar framework.

node yields:

$$P \approx 4t_0\frac{\delta}{l} + 8AE\frac{\delta^3}{l^3} \tag{6.4}$$

with t_0 the prestressing force and AE the axial stiffness. Thus, in absence of prestress, the framework has zero vertical stiffness in the initial configuration. For a small deflection δ the stiffness is proportional to δ^2, but with prestress the stiffness is proportional to the level of prestress and the mechanism is stabilized.

In recent years, more additions were made to Maxwell's rule based on symmetry considerations by Fowler and Guest [9].

6.2 Linear structural analysis

This section will recapitulate the linear structural analysis of pin-jointed bar frameworks, and will frequently refer to work by Pellegrino and Calladine [20] for detailed analysis of the equilibrium equations.

Traditional linear structural analysis requires three principles to be satisfied; that internal forces t are in equilibrium with the applied loads f, that any internal deformation e is compatible with external displacements d, and that internal forces t and elongations e are related by a material law.

For small perturbations about the initial equilibrium configuration of a structure, and under the absence of internal prestress (so without zero-free-length springs), these relationships can be linearized as three matrix relationships

$$\mathbf{f} = \mathbf{A}\mathbf{t} \tag{6.5}$$

$$\mathbf{e} = \mathbf{C}\mathbf{d} \tag{6.6}$$

$$\mathbf{t} = \mathbf{G}\mathbf{e} \tag{6.7}$$

where $\mathbf{A}$ is known as the equilibrium matrix, $\mathbf{C}$ as the compatibility matrix and $\mathbf{G}$ a diagonal matrix with element stiffnesses $\frac{EA}{L}$ on the diagonal. The solution in a problem in structural analysis requires the simultaneous solution of those equations.

6.2.1 Static-kinematic duality

First, the static-kinematic duality of the above equations will be explained, which will result in showing that $\mathbf{A}^T = \mathbf{C}$. This is done via the virtual work principle, where internal and external work are equated

$$\delta W_{int} = \delta W_{ext}$$

$$\delta\mathbf{e}^T\mathbf{t} = \delta\mathbf{d}^T\mathbf{f} \tag{6.8}$$

and when substituting equations 6.5 and 6.6 this becomes

$$\delta\mathbf{d}^T\mathbf{C}^T\mathbf{t} = \delta\mathbf{d}^T\mathbf{A}\mathbf{t} \tag{6.9}$$

which is valid for *any* displacement $\delta\mathbf{d}$, and results in

$$\mathbf{C}^T = \mathbf{A}. \tag{6.10}$$

Due to this static-kinematic duality, the analysis of the equilibrium equations of a structure in order to identify the states of self-stress present in a statically indeterminate structure, will also show the presence of inextensional mechanisms in a kinematically indeterminate structure, Pellegrino and Calladine [20].

6.2.2 Linear stiffness matrix

Commonly, using the Stiffness Method of structural analysis, the internal forces are condensed out and the three sets of equations 6.5–6.7 are combined to form a single stiffness relationship which relates external forces with nodal displacements,

$$\mathbf{f} = \mathbf{A}\mathbf{t}$$

when substituting 6.7, 6.5 and 6.10 we obtain

$$\mathbf{f} = \mathbf{A}\mathbf{G}\mathbf{e} \tag{6.11}$$

$$\mathbf{f} = \mathbf{A}\mathbf{G}\mathbf{C}\mathbf{d} = \mathbf{A}\mathbf{G}\mathbf{A}^T\mathbf{d} = \mathbf{K_0}\mathbf{d} \tag{6.12}$$

where $\mathbf{K_0}$ is referred to as the *linear stiffness matrix*, which is solely based on linear relations between displacements, elongations and applied forces.

If a system is to be stable, the matrix $\mathbf{K}_0$ must be positive definite or, equivalently, of full rank, i.e. $r(\mathbf{K}_0) = 3j - c$. It will be shown, however, that determining the rank of the equilibrium matrix is sufficient to establish the stability of the system [8].

$$\mathbf{K}_0 = \mathbf{A}\mathbf{G}\mathbf{A}^T = \mathbf{A}(\mathbf{G}^*\mathbf{G}^{*T})\mathbf{A}^T = (\mathbf{A}\mathbf{G}^*)(\mathbf{A}\mathbf{G}^*)^T \tag{6.13}$$

where $\mathbf{G}^*$ is also a diagonal matrix whose diagonal elements $s_k^* = \sqrt{s_k}$. Since for regular bar frameworks all elements from $\mathbf{G}$ are positive, the rank of $\mathbf{A}\mathbf{G}^*$ takes it value from the rank of $\mathbf{A}$. Furthermore, since the rank of a matrix $\mathbf{X}$ is the same as that of $\mathbf{X}\mathbf{X}^T$ then

$$r(\mathbf{K}_0) = r[(\mathbf{A}\mathbf{G}^*)(\mathbf{A}\mathbf{G}^*)^T] = r(\mathbf{A}\mathbf{G}^*) = r(A) = r. \tag{6.14}$$

Hence, whether $\mathbf{K}_0$ is positive definite can be judged from the rank of equilibrium matrix $\mathbf{A}_{3j-c\times b}$. This will be elaborated on in section 6.2.4.

6.2.3 Static and kinematic indeterminacy

Before starting with the matrix analysis of the equilibrium matrix **A** the concepts of static and kinematic indeterminacy are recapitulated, as they will contribute to the understanding of the analysis as developed in Pellegrino and Calladine [20] and expanded in Pellegrino [18].

Structures are referred to as *statically determinate* if there is a unique solution to the equilibrium equations for any applied loading. If a structure is *statically indeterminate*, then it will admit different states of self-stress, where the structure can be stressed against itself, even in the absence of external loads.

Structures are referred to as *kinematically determinate* if there is a unique solution to the compatibility equations for any set of internal extensions. If a structure is *kinematically indeterminate*, then there will be certain movements of the joints, where, at least to the first-order approximation, there are no changes in bar lengths. This is referred to as a mechanism. Mechanisms will be discussed in greater detail in section 6.2.6.

6.2.4 Matrix analysis of equilibrium matrix

Fundamentally speaking, the $(3j - c \times b)$ matrix **A** can be considered a linear operator between two vector spaces, the bar space $\Re^b$ and the joint space $\Re^{3j-c}$. The four fundamental subspaces associated with **A** are shown in table 6.1 and provide a great deal of information about the structure. For a recapitulation of vector subspaces the reader is referred to a Linear Algebra book such as Lay [14].

Note that the dimensions of the four subspaces can easily be computed once the rank r of the equilibrium matrix is known.

Nullspace of A

The nullspace of **A** contains all tensions t that are solutions of $\mathbf{At} = \mathbf{0}$; in other words they are internal stresses under absence of external loads. If the nullspace of **A** has dimension $s = 0$, the assembly admits no sets of self-equilibrated tensions and it is therefore *statically determinate*. If $s > 0$ the assembly is *statically indeterminate* and $s = b - r$ is the number of independent states of self-stress it admits.

Left-nullspace of A

If the left-nullspace of **A** has dimension $m = 0$, any load can be equilibrated by the assembly in its configuration. Because the left-nullspace of **A** coincides with the nullspace of **C**, any set of displacements **d** is thus extensional, and therefore associated with a unique set of compatible elongations. Assemblies with $m = 0$ are known as *kinematically determinate*.

		Equilibrium **A**		Compatibility **C**
Bar space $\Re^b$	r	Rowspace: bar tensions in equilibrium with loads in the column space	=	Column space: compatible bar elongations
		$\perp$		$\perp$
	s=b-r	Nullspace: states of self-stress. (Solutions of $\mathbf{At} = \mathbf{0}$)	=	Left-nullspace: incompatible bar elongations
Joint space $\Re^{3j-c}$	r	Column space: loads which can be equilibrated in the initial configuration.	=	Row space: extensional displacements.
		$\perp$		$\perp$
	m=3j-c-r	Left-nullspace: loads which cannot be equilibrated in the initial configuration.	=	Nullspace: inextensional displacements. (Solutions of $\mathbf{Cd} = \mathbf{0}$)

Table 6.1: Four fundamental subspaces associated with the equilibrium matrix **A** and the compatiblity matrix **C**. The = sign indicates the two subspaces coincide, while $\perp$ indicates that they are orthogonal complements of one another. Adapted from Pellegrino [18].

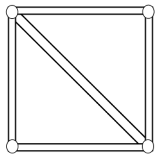
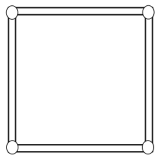
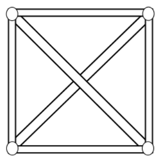
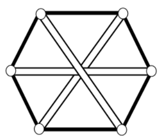
Structural type		Static and Kinematic properties	2D example
I	$s = 0$; $m = 0$	Statically determinate and kinematically determinate	
II	$s = 0$; $m > 0$	Statically determinate and kinematically indeterminate	
III	$s > 0$; $m = 0$	Statically indeterminate and kinematically determinate	
IV	$s > 0$; $m > 0$	Statically indeterminate and kinematically indeterminate	

Table 6.2: Four different types of structures.

If $m > 0$ the assembly is *kinematically indeterminate* and $m = 3j - c - r$ is the number of independent inextensional mechanisms, which are spanned by the nullspace of $\mathbf{C}$. This means that those displacements do not, at least to the first order, require any forces (and consequently energy) and are thus inextensional.

Structural types

From the foregoing discussion, it would make sense to introduce four types of structures, as seen in table 6.2, depending on their static or kinematic (in)determinacy. Tensegrity structures are of structural type IV and are thus both kinematically and statically indeterminate [18].

Singular Value Decomposition

The value of r can be determined in a number of ways, but the use of Singular Value Decomposition (SVD) [19] on the equilibirum matrix would also give orthogonal sets of m inextensional mechanisms and s states of self-stress, as follows

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \tag{6.15}$$

where $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{3j-c}\}$ consists of a set of left singular vectors, $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_b\}$ contains a set of right singular vectors, and a set of singular values is found in the first r non-zero diagonal elements of $\boldsymbol{\Sigma}$.

The singular vectors, all of unit norm, can be grouped into the following submatrices

$$\begin{aligned} \mathbf{U}_r &= \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\} \qquad & \mathbf{U}_m &= \{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_{3j-c}\} \\ \mathbf{V}_r &= \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \qquad & \mathbf{V}_s &= \{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_b\} \end{aligned} \tag{6.16}$$

which have the following interpretations:

$\mathbf{U}_r$ contains modes of extensional deformation (i.e. loads that can be equilibrated by the structure in its current configuration);

$\mathbf{U}_m$ contains modes of inextensional deformation, i.e. mechanisms (i.e. loads that cannot be equilibrated by the structure in its current configuration);

$\mathbf{V}_r$ contains sets of kinematically compatible extensions corresponding, through the singular values, to the extensional modes in $\mathbf{U}_r$;

$\mathbf{V}_s$ contains sets of kinematically incompatible extensions (i.e. states of self-stress).

The subspaces spanned by $\mathbf{U}$ and $\mathbf{V}$ have dual statical and kinematical interpretations because the equilibrium and compatibility matrices are transposes of each other. The SVD function is readily available in programs such as MATLAB.

Henceforth the basis for the states of self-stress is referred to as

$$\mathbf{SS} = [\mathbf{v}_{r+1}, \ldots, \mathbf{v}_b] \tag{6.17}$$

and the base for the mechanisms as

$$\mathbf{D} = [\mathbf{u}_{r+1}, \ldots, \mathbf{u}_{3j-c}]. \tag{6.18}$$

This notation is adopted for compatibility with notations in literature.

6.2.5 Rigid-body mechanisms

The mechanisms in $\mathbf{D}$ can either be internal mechanisms or rigid-body mechanism, as a result of inadequate kinematic constraints on the structure. A scheme to separate the internal mechanisms from the rigid-body ones was proposed by Pellegrino and Calladine [20], and it can cope with up to six rigid-body mechanisms.

Any rigid-body displacement in three-dimensional space may be described by a translation $\mathbf{n}$ and a rotation $\mathbf{r}$. A displacement $\mathbf{d}_i$ of a point i in such a rigid-body motion is given by:

$$\mathbf{d}_i = \mathbf{n} + \mathbf{r} \times \mathbf{p}_i \tag{6.19}$$

where $\mathbf{p}_i$ is the position vector of point i in the original configuration. If the structure has a total of c constrained degrees of freedom, the system of c equations in six unknowns is

$$\mathbf{R} \begin{pmatrix} \mathbf{n} \\ \mathbf{r} \end{pmatrix} = \mathbf{0} \tag{6.20}$$

where $\mathbf{R}$ is a matrix of size $c \times 6$ and whose rank $r_{\mathbf{R}}$ counts the number of kinematic constraints that suppress the rigid-body degrees of freedom. Thus the number of rigid-body mechanisms is

$$m_{rb} = 6 - r_{\mathbf{R}}. \tag{6.21}$$

Now those rigid-body motions have to be removed from the matrix $\mathbf{D}$, which could be done by the Gram-Schmidt orthogonalization procedure [14] or by for instance MATLAB's QR-decomposition. This results in a set of $m - m_{rb}$ independent internal mechanisms.

6.2.6 Stability of mechanisms

The fact that a kinematically indeterminate structure has an internal mechanism does not imply that this motion has no stiffness, as long as the structure is stressed, either through self-stress or external loads.

If the structure eventually tightens up as the mechanism is displaced, i.e. the mechanism involves higher than first-order changes in bar length, the mechanism is called *infinitesimal*. Otherwise the mechanism is called *finite*. Infinitesimal mechanisms of a structure can be classified depending on the order to which the changes of bar length relate to the displacements. Vassart et al. [29] devised a method to find the order of those mechanisms and to identify finite mechanisms.

First-order infinitesimal mechanisms – mechanisms associated with second-order changes of bar lengths – may be stabilized by a state of self-stress, and the structure becomes prestress stable. However, in some cases not all first-order mechanisms can be stabilized by a single state of self stress. Additionally, second-and-higher order and finite mechanisms can never be stabilized by a state of self-stress.

The rigidity/stability of mechanisms is also discussed in mathematical Rigidity Theory (e.g. Connelly and Whiteley [4]), with slightly different terminology. A brief overview of this approach will be provided in section 6.4.

Product Forces

In order to find the stiffness of a structural mechanism, Pellegrino [18] used a method which detects the presence of first-order rigidity under self-stress, by making use of Product Forces.

The approach is motivated by the explanation that a structure has two distinct ways of equilibrating an additional load $\delta\mathbf{f}$. It can (i) alter its bar tensions by $\delta\mathbf{t}$ and hence carry the load with little displacement from its initial configuration; or (ii) deform inextensionally at approximately constant stress, in which case the load is equilibrated by out-of-balance forces arising from the change of geometry. In many cases the assembly responds by a combination of modes (i) and (ii).

The first mode is accounted for by the equilibrium equation 6.5 and the second is often referred to as the *geometric stiffness* in non-linear FEA. The mechanisms are in the nullspace of the equilibrium matrix $\mathbf{A}$, so in order to find the stiffness, attention has to be turned to mode (ii).

First, any inextensional mode is given by

$$\mathbf{d} = \mathbf{D}\beta \tag{6.22}$$

where $\beta = \{\beta_{r+1}, \ldots, \beta_{3j-c}\}^T$ is a vector of combination coefficients.

Next, the product forces are defined which represent the out-of-balance force resulting from imposing a mechanism on an assembly with a state of self-stress.

For each state of self-stress $\mathbf{t}_i$ and each mechanism $\mathbf{d}_j$ a vector of geometric loads $\mathbf{g}_{ij}$ can be computed as

$$\mathbf{g}_{ij} = \mathbf{B}_{\mathbf{d}_j}\mathbf{t}_i \tag{6.23}$$

where $\mathbf{B}_{\mathbf{d}_j}$ is similar to the equilibrium matrix $\mathbf{A}$ but with coefficients of type $(d^j_{kx} - d^j_{lx})/l_{kl}$, i.e. displacements of mechanism j. The m geometric loads for self-stress state i form the columns of the geometric load matrix $\mathbf{G}_i$ of size $3j - c \times m$. Each state of self-stress has its own matrix $\mathbf{G}_i$.

Given these equations, the criterion for the stabilisation of internal mechanisms, as developed by Pellegrino, resulted in:

$$\beta^T \left[\sum_{i=1}^{s} \mathbf{G}_i^T \mathbf{D} \alpha_i \right] \beta$$

where α_i are combination coefficients for the state of self-stress. This criterion equation has to be positive definite for there to be positive stiffness for any mechanism of the structure. If the criterion is merely positive semi-definite, structures such as a collinear parallellogram would be classified as rigid [5].

6.3 Non-linear/prestressed FEA

As was shown in the previous section, in order to analyse the stability of internal mechanisms the stiffening effects of prestresses have to be taken into account. To achieve this, Pellegrino [18] made use of the *product forces* method as descibed in 6.2.6. Recent literature [8, 11, 17] has shown that this method is actually a special case of the geometric stiffness matrix of the *geometrically non-linear tangent stiffness matrix* found in non-linear FEA.

This makes non-linear FEA the next step in the stability analysis of pin-jointed structures, and it now takes into account all aspects of the pin-jointed structure; topology, geometry, material properties and prestresses. The key part of the non-linear analysis is the *tangent stiffness matrix* $\mathbf{K}_t$ that incorporates all these factors.

Guest [11] shows a novel derivation of the tangent stiffness matrix of a pin-jointed bar framework, which unites several different formulations and which yields some interesting insight into the use of zero-free-length springs in structures.

6.3.1 Prestressed FEA - modified axial stiffness

In Guest [11] the tangent stiffness matrix is derived for a *prestressed single bar* floating in space (see Figure 6.3) with external forces $\mathbf{f}_1$ and $\mathbf{f}_2$ at the nodes and internal bar tension t. After deriving the equilibrium equations with respect to

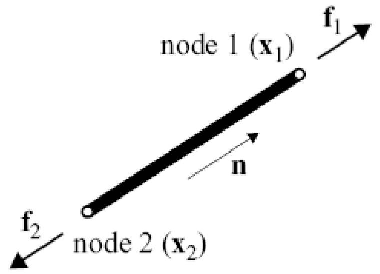


Figure 6.3: A single bar floating in space, connected by two nodes.

the nodal coordinates, the following tangent stiffness equation is obtained for a single bar

$$\mathbf{K}_s = \mathbf{a}_s[\hat{g}]\mathbf{a}_s^T + \mathbf{S}_s \tag{6.24}$$

where $\mathbf{a}_s$ is the *equilibrium matrix* for a single bar,

$$\mathbf{a}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} \tag{6.25}$$

relating bar tension and nodal force,

$$\mathbf{a}_s[t] = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{6.26}$$

and $\mathbf{S}_s$ is the *geometric stiffness matrix*, or *stress matrix* for a single bar

$$\mathbf{S}_s = \hat{t}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}. \tag{6.27}$$

The most interesting aspect of this derivation, however, is the variable $\hat{g}$ known as the *modified axial stiffness.* It is defined as:

$$\hat{g} = g - \hat{t} \tag{6.28}$$

where g is the axial stiffness, for axially loaded bars $\frac{EA}{l_0}$, and $\hat{t}$ the tension coefficient, defined as:

$$\hat{t} = \frac{t}{l} \tag{6.29}$$

with t the tension in the element and l the length of the bar.

6.3.2 Zero-free-length springs

The introduction of the modified axial stiffness will lead to an important conclusion when applied to zero-free-length springs. Guest has shown that the following equality holds

$$\frac{d\hat{t}}{dl} = \frac{\hat{g}}{l}, \tag{6.30}$$

and when realising that for zero-free-length springs the tension coefficient $\hat{t}$ is equal to the *constant* spring stiffness K_{zfl}, the modified axial stiffness returns zero!

This important conclusion was not immediately intuitive from conventional formulations of the tangent stiffness matrix and has great consequences for incorporating zero-free-length springs in structural analysis. Because the modified axial stiffness vanishes, so does the first part of equation 6.24 and only the stress matrix remains in the tangent stiffness of a zero-free length spring. In other words, the geometry is of no influence when using zero-free-length springs, only internal forces and connectivity matter! When looking at this conclusion from a static balancing point of view, it makes sense, because such systems remain in equilibrium by their choice of springs and topology, regardless of their exact nodal positions, i.e. geometry.

The consequences for existing analysis methods are significant, and become obvious when looking at the newly derived tangent stiffness matrix of the total structure

$$\mathbf{K}_t = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^T + \mathbf{S} = \mathbf{K}_{\hat{0}} + \mathbf{S} \tag{6.31}$$

where $\mathbf{A}$ is the *equilibrium matrix* for the complete structure. Suddenly the observation that the rank of $\mathbf{A}$ is directly coupled to the rank of the modified material stiffness matrix $\mathbf{K}_{\hat{0}}$ no longer holds, because the matrix $\hat{\mathbf{G}}$ is no longer positive definite.

Research will have to show how the analysis of the equilibrium matrix as discussed in section 6.2, will interact with and complement the results from the tangent stiffness matrix with zero-free-length springs. One would for instance have to check whether the zero-stiffness modes found with the analysis of equilibrium-matrix (infinitesimal, prestress stable or finite ones) do not coincide with the zero-stiffness modes introduced due to the use of the zero-free-length springs (see Figure 6.4).

6.3.3 Zero-stiffness modes

Once the tangent stiffness matrix $\mathbf{K}_t$ of a structure is known, finding the presence of zero-stiffness modes becomes trivial; the eigenvalues and -modes of the matrix represent the modal stiffnesses with their corresponding displacements:

$$\begin{aligned} \mathbf{K}_t\mathbf{u} &= \lambda\mathbf{u} \\ &= \mathbf{f}, \end{aligned} \tag{6.32}$$

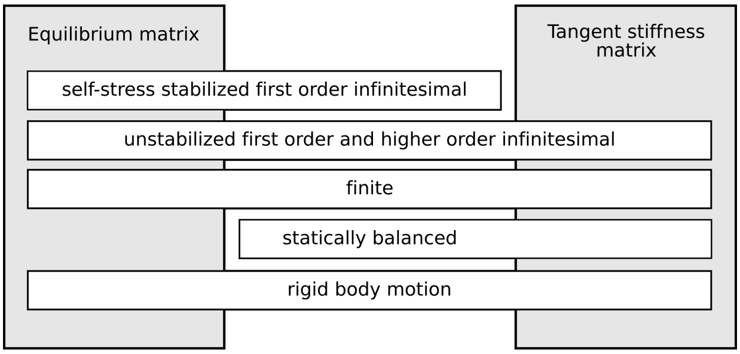


Figure 6.4: Zero-stiffness modes found in equilibrium matrix and tangent stiffness matrix.

so when considering that a stiffness is defined by $\frac{\mathbf{f}}{\mathbf{u}}$ the eigenvalue λ can be considered the modal stiffness. Assuming that the tensegrity is completely unconstrained in 3D space, there will be at least 6 zero-stiffness modes, corresponding to the rigid-body motions.

If there are more than 6 zero-stiffness modes, the rigid-body motions have to be removed as was shown in section 6.2.5. The remaining independent zero-stiffness modes may now correspond to either statically balanced modes or higher-order infinitesimal and finite mechanisms. At this point it is not completely clear to the author how to distinguish between these two possible aspects.

Deng and Kwan [8] mention the possibility that the tangent stiffness matrix contains certain zero-stiffness modes (or in their words, $\delta\mathbf{d}^T(\mathbf{K}_0 + \mathbf{K}_g)\delta\mathbf{d} = 0$ for some $\delta\mathbf{d}$). They state that these systems are in critical stability and require investigation of higher-than-second-order variations of the potential energy in the system, with consequent computational difficulties.

6.3.4 Classic non-linear FEA

Due to the fundamental importance of the above insight in the study of neutral-stability structures with zero-free-length springs, it was deemed relevant to verify the newly derived tangent stiffness matrix via another route: the classic non-linear FEA. A complete derivation of this comparison can be found in Appendix A.

The comparison concludes that, after some rewriting, the classic non-linear FEA (e.g. Crisfield [7]) produces the following tangent stiffness matrix, consisting of three individual matrices; the first is the *linear stiffness matrix*, the other two form the *geometric stiffness matrix*

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t\sigma 1} + \mathbf{K}_{t\sigma 2} \tag{6.33}$$

with the following terms

$$\mathbf{K}_{t1} \quad = \quad \mathbf{a}_s[g]\mathbf{a}_s^T \tag{6.34}$$

$$\mathbf{K}_{t\sigma 1} = \hat{t}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix} \tag{6.35}$$

$$\mathbf{K}_{t\sigma 2} = -\mathbf{a}_s[\hat{t}]\mathbf{a}_s^T. \tag{6.36}$$

When $\mathbf{K}_{t1}$ and $\mathbf{K}_{\sigma 2}$ are combined into $\mathbf{K}_{\hat{0}}$, it is easy to see that they lead to the same formulation as equation 6.24, confirming the analysis.

6.4 Rigidity Theory

Equivalent formulations to the structural analysis discussed in previous sections, can also be found in mathematical Rigidity Theory, such as Connelly and Whiteley [4]. Publications in that field of science approach the rigidity of frameworks from a very different perspective, which is not as directly linked to physical realisation as classic structural analysis.

A configuration of n ordered points in D-dimensional space is denoted by

$$\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n]. \tag{6.37}$$

Next a *tensegrity framework* $G(\mathbf{P})$ is defined, by a graph G on $\mathbf{P}$ where the edges are designated as either a cable, a strut or a bar; cables cannot increase in length, struts cannot decrease in length and bars cannot change length.

A broadly defined term *rigidity* is provided; it incorporates all frameworks which do not have finite mechanisms. It is the weakest reasonable notion of rigidity, and is based on the question whether or not there are only trivial flexes of $G(\mathbf{P})$. A linearization of rigidity gives the term *infinitesimal rigidity*, which is equivalent to the *first-order rigidity* found with the linear equilibrium matrix $\mathbf{A}$ in classic structural analysis.

Subsequently, a stress ω_{ij} is defined for each member of the graph G. The stress in rigidity theory is defined mathematically, regardless of material properties of the framework. A stress ω is a proper stress if $\omega_{ij} > 0$ for a cable and $\omega_{ij} < 0$ for a strut. Note that the stress for a bar can be either positive, negative or zero. The thus-defined mathematical stress is equivalent to the *force-density* as discussed in regular structural engineering.

Every $G(\mathbf{P})$ is *statically rigid* if every equilibrium load can be resolved by a proper stress, where equilibrium load is defined as a load that does not result in a rigid-body motion. Static rigidity is equivalent to first-order rigidity.

When there is always a unique solution to how loads are resolved in a system, it is statically determinate, or *isostatic* in mathematical terms. Many frameworks are not isostatic and in order to know how loads are distributed, material properties are needed. However, for answering the yes-or-no question, whether the framework is rigid/not-rigid, one does not.

The next natural step is to introduce a form of energy in the system or potential function that provides "stability" to a given framework. This need not be a physical energy, but a purely mathematical function $\mathbf{H}(\mathbf{P})$ should suffice. If

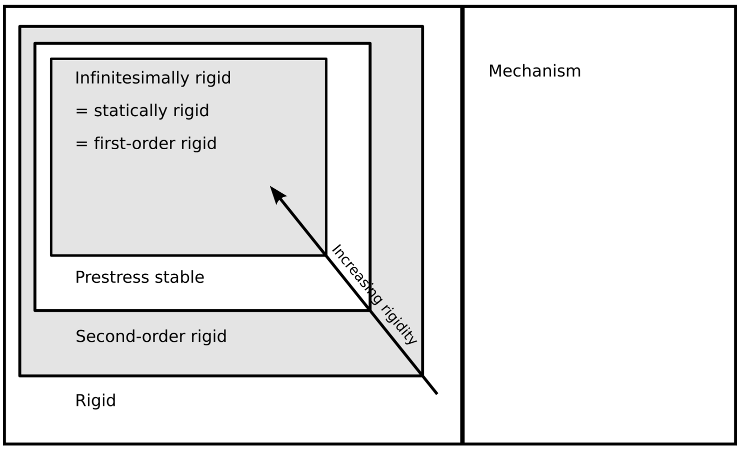


Figure 6.5: Classes of rigidity as defined in Rigidity Theory.

for all configurations $\mathbf{Q}$ sufficiently near $\mathbf{P}$, $\mathbf{H}(\mathbf{Q})$ has a strict minimum at $\mathbf{P}$, modulo rigid motions, then $G(\mathbf{P})$ is rigid. In this situation, the stress ω_{ij} is seen as a first-order derivative of the energy.

Now a rigidity matrix $\mathbf{R}(\mathbf{p})$ is defined, which is equivalent to the compatiblity matrix 6.6, but also takes into account higher-order variations. Furthermore, a *stress matrix* $\boldsymbol{\Omega}$ is defined, and the Hessian (second order derivative) of the energy $\mathbf{H}$ is given as

$$\boldsymbol{\Omega} + \mathbf{R}(\mathbf{p})^T \mathbf{D} \mathbf{R}(\mathbf{p}) = \mathbf{S} \tag{6.38}$$

which is equivalent to the tangent stiffness matrix formulation found earlier, with $\boldsymbol{\Omega}$ being equivalent to the geometric stiffness matrix. The matrix $\mathbf{D}$ is diagonal with elements d_{ij} that are derivatives of the stresses ω_{ij}; this means they are equivalent to tension-coefficient derivatives, compare equation 6.30.

The above can intuitively be compared with energy stored in a spring, $E_{spring} = 1/2ku^2$. The first derivative would provide ku which is a force, which is equivalent to the stress ω in the framework. A second derivative would provide the spring stiffness k, and this holds for frameworks as well, where the second-order derivative provides the tangent stiffness matrix.

With the above tools, structures are defined *prestress stable* if the ω are proper, $\mathbf{S}$ is positive semi-definite with only the trivial infinitesimal flexes (rigid-body motions). It is also shown that if there is a stabilizing self-stress for one set of positive stiffness coefficients, then, for any other positive stiffness coefficients there is a stabilizing self-stress as well. Question is whether this also holds for zero-valued stiffness coefficients, as is the case with zero-free-length springs.

Finally *second-order rigidity* is defined, which is equivalent to saying that for each possible loading of the framework, there is a self-stress that stabilizes the framework for at least this loading. However, if the framework is not pre-stress stable, different loadings will require different distributions of the tensions and compressions in the members.

Concluding, mathematical Rigidity Theory provides another insight into structural analysis with a whole new set of tools, and the search for zero-stiffness modes may also be continued along these lines.

6.5 Discussion

This chapter has discussed the stability of pin-jointed bar frameworks, ranging from Maxwell's rule to full geometrically non-linear FEA. In the process, all aspects that influence the stability of a structure have been included: geometry and topology, element properties and internal forces (see figure 6.6).

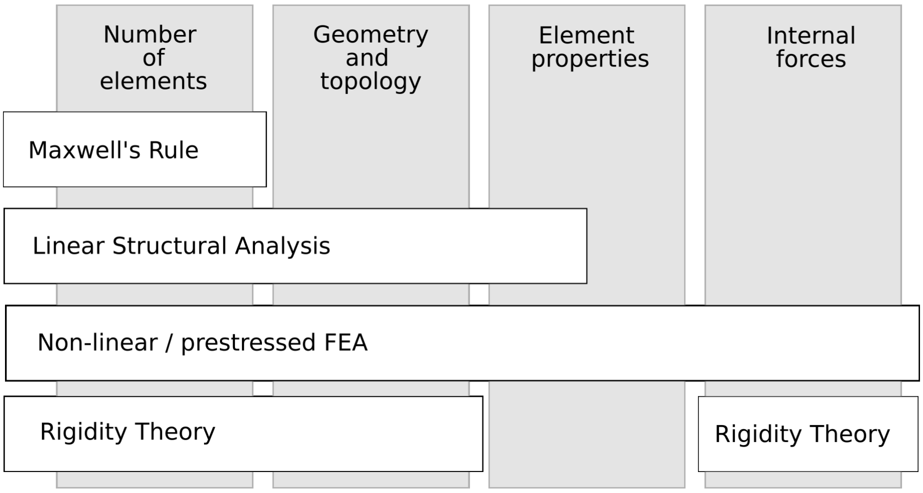


Figure 6.6: Overview of structural analysis methods.

An important question to be answered was how zero-free-length springs could be incorporated in this analysis and how zero-stiffness modes would manifest themselves. It was shown that in order to incorporate zero-free-length springs into the structural analysis, a tangent stiffness matrix has to be drawn up.

An important insight was provided by a novel derivation of the tangent stiffness matrix, which introduced the modified axial stiffness. For zero-free-length springs, the modified axial stiffness vanishes (and consequently the modified material stiffness matrix) and only the stress matrix remains. In other words, the geometry is of no influence when using zero-free-length springs, only internal forces and connectivity matter.

Due to the vanishing of the linear stiffness matrix for zero-free-length springs, the relevance of the regular linear structural analysis approach which analyses the equilibrium matrix $\mathbf{A}$ has to be reconsidered. One will have to check whether zero-stiffness modes found are a result of the presence of the zero-free-length springs, or result of structural geometry.

Furthermore, when analysing zero-stiffness modes of the tangent stiffness matrix, it must be considered that they might also be the result of higher-order or finite mechanisms. A method is needed to separate these from the zero-stiffness modes resulting from the zero-free-length springs. Intuitively this could be done by first analysing the structure with the same internal forces, but with different material properties. If the structure does not have zero-stiffness modes in that configuration, then any modes found in the configuration with zero-free-length springs would be statically balanced ones.

Another aspect to study in future research is the question whether a statically balanced zero-stiffness mode is not merely infinitesimal, but also valid over a larger range of motion. Intuitively, one would say that because the effects of the zero-free-length springs is independent of actual geometrical position, chances are good that a zero-stiffness mode will remain statically balanced throughout a larger range of motion. This will have to be verified formally.

One interesting aspect that can be considered an interesting path of research is that if a zero-stiffness mode coincides with an affine transformation, as defined in chapter 5, it will very likely remain balanced over a longer range, as a tensegrity will remain in equilibrium under affine transformations.

One final remark is that the stress matrix $\mathbf{\Omega}$ is equivalent to the matrix $\mathbf{D}$ from the force-density form-finding method, which links that method to the structural analysis.

The above discussion should provide several interesting routes of research with regard to zero-free-length springs and zero-stiffness modes.

Chapter 7

Load Analysis

The third step in the design and analysis of tensegrity structures is the load analysis; how does the structure react to external loading. Non-linear FEA is deemed the best analysis method, and some computational methods, suitable for (statically balanced) tensegrities, for solving the non-linear equations are very briefly reviewed.

7.1 Analysis methods

For simple structures load analysis can be combined with the form-finding method with reduced coordinates to easily obtain solutions to the external loading Crane III et al. [6]. For more complex structures numerical methods have to be considered.

The Force Method as mentioned by Pellegrino [18] and Tibert [27] is only useful for small linear displacements, and even then Pellegrino [18] proposes a number of corrective measures for certain types of structures such as tensegrities. It may provide more insight than non-linear FEA, but is insufficient for use with zero-free-length springs as the non-linear properties are of greatest importance.

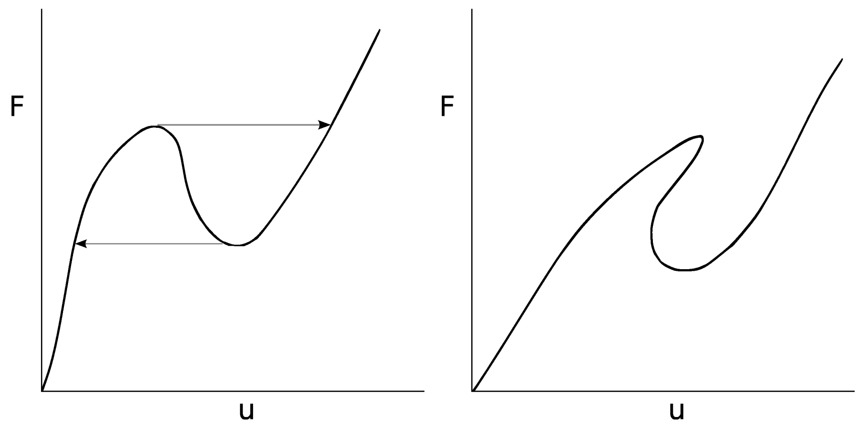
The next obvious step would be to work with standard non-linear FEA, which takes into account all the geometric non-linearities as discussed in the previous chapter. A generic solver for tensegrities will also have to take into account the possibility of slacking cables, although if all cables are replaced by zero-free-length springs, they will never be slack. For completeness' sake, the physical free-length of the spring, and buckling loads of the bars have to be taken into consideration as well.

The mentioned analysis methods are based on the structural analysis discussed in previous chapters. One could also consider a multibody dynamics approach if dynamic properties of the system are deemed relevant, and where zero-free-length springs are inserted as passive elements. However, at this time the author believes that approach will not yield more insight than FEA.

7.2 Computational methods

To obtain an overview of the computational methods to efficiently and reliably solve the non-linear equations for the deflection of a loaded structure is worthy of a report in itself, so this section will only very briefly recapitulate some relevant solution techniques, as found for instance in Van Keulen [28] or Crisfield [7].

The main point of non-linear equations, is that they are only valid at a certain point on the equilibrium path. So, whenever a small force perturbation is made, the corresponding small displacement is calculated, but the original tangent stiffness matrix is no longer valid at the new position and will have to be recalculated. This is often done iteratively to reduce errors, such as with the *(modified) Newton-Raphson* method, or the *incremental-iterative* method.



(a) Limit points. (b) Snapback.

Figure 7.1: Typical non-linear responses in FEA.

Common difficulties encountered during non-linear FEA include the presence of limit points and snapbacks (see Figure 7.1). These are very likely to occur in tensegrities, as due to their sometimes delicate balance, they may suddenly jump to new equilibrium positions.

Additionally, for zero-stiffness modes the tangent stiffness matrix is singular and unique solutions are not always possible. Furthermore, a small increase in displacements would yield zero external forces in that case, making new increments impossible.

For both these reasons, computational methods such as *steering equations* and *arc-length* methods will have to be considered when investigating the static balancing over a larger range of motion, as they do not entirely rely on increments in external force or displacements.

7.3 Discussion

The topic of load analysis and suitable solutions methods for non-linear FEA have only very briefly been discussed, and it is suggested to study the topic in greater detail when required. For the time being, *steering equations* or the *arc-length method* seem the best potential computational methods, due to the likely presence of limit points and snapbacks, and the presence of zero-stiffness modes.

Chapter 8

Results and conclusion

This section will recapitulate the results and conclusions from previous chapters, as well as indicate several interesting approaches and unanswered questions in the study of statically balanced tensegrity structures.

8.1 Results

- Zero-stiffness or neutral stability is the key feature of static balancing, and provides a link with stability analysis of structures;
- In order to find more complex examples of statically balanced tensegrities, methods other than the existing modification rules are needed;
- Tensegrity structures are a subset of pin-jointed bar frameworks and therefore use can be made of existing design and analysis tools for those structures;
- Design and analysis of tensegrities can be split up into form-finding, structural stability and loadal analysis;
- Existing form-finding methods are not quite sufficient, due to lack of possible boundary conditions;
- Affine transformations of existing tensegrities are an interesting concept, and may hold a clue to static balancing;
- Structural analysis of tensegrities requires geometrically non-linear FEA analysis, due to the presence of prestresses;
- Introduction of zero-free-length springs in structural analysis has the surprising result that the modified material stiffness matrix drops out, and only the stress matrix remains. This means that for static balancing the geometry of the structure is less important than the internal stresses and element connectivity;

- Zero-stiffness modes in the tangent stiffness matrix should be analysed to determine if they are higher-order infinitesimal or finite mechanisms, or truly statically balanced modes;
- Due to the presence of zero-stiffness modes and possible occurences of limit points and snapbacks in the equilibrium path of tensegrities, steering equations and arc-length methods might be suitable computational methods for solving the non-linear FEA equations.

8.2 Future approaches

After having provided a solid theoretical foundation for the study of statically balanced tensegrity mechanisms, some interesting and possibly fruitful research paths can be identified. These will briefly be listed here:

- Analyse existing tensegrity structure by adding zero-free-length springs, and analysing possible zero-stiffness modes;
- Theoretical analysis of how to distinguish between statically balanced zero-stiffness modes and other zero-stiffness modes, such as higher-order infinitesimal mechanisms and finite mechanisms;
- Exploration of the concept of affine transformations and how they might be linked to static balancing.

Generally speaking these steps are aimed at the structural analysis of the structures. Form-finding and load analysis will be left untouched for the time being, as it is the author's impression that these aspects are only useful to investigate further once more fundamental insight is gained by the study of structural stability of structures with zero-free-length springs.

Appendix A

Comparison modified axial stiffness and non-linear FEA

A.1 Introduction

The objective of this appendix is to provide a different route to the geometrically non-linear tangent stiffness matrix found in Guest [11] and to identify somof the underlying assumptions in its derivation.

Guest's work introduces the concept of a 'modified axial stiffness' in the formulation of the tangent stiffness matrix for pin-jointed structures. This variable provides valuable (and otherwise not immediately intuitive) insight into the special case of introducing zero-free-length springs [12] to structures.

Due to the fundamental importance of this step in the study of neutral-stability structures with zero-free-length springs it is deemed relevant to verify the resulting tangent stiffness matrix via another route: the classic non-linear FEA.

Literature on non-linear FEA is known to be meticulous in keeping track of deformed and undeformed lengths during the creation of the tangent stiffness matrix. This will lead to a more complex formulation, from which it is possible to establish which assumptions were used in Guest [11].

A.2 Modified axial stiffness

Figure A.1 shows a free body diagram of a bar in three-dimensional space. The external forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are in equilibrium with the internal bar tension t, and the nodes have position vectors, $\mathbf{x}_1$ and $\mathbf{x}_2$.

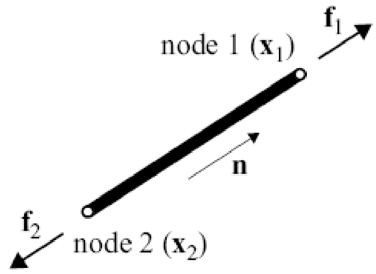


Figure A.1: A single bar floating in space, connected by two nodes.

The equilibrium at the two nodes 1 and 2 can be expressed in terms of the bar tension t and the unit vector $\mathbf{n} = (\mathbf{x}_1 - \mathbf{x}_2)/l$,

$$\mathbf{f}_1 = \mathbf{n}t \tag{A.1}$$

$$\mathbf{f}_2 = -\mathbf{n}t. \tag{A.2}$$

Alternatively, with a *tension coefficient* or *force density* defined as $\hat{t} = t/l$, the equilibrium can be rewritten as

$$\mathbf{f}_1 = (\mathbf{x}_1 - \mathbf{x}_2)\hat{t} \tag{A.3}$$

$$\mathbf{f}_2 = (-\mathbf{x}_1 + \mathbf{x}_2)\hat{t}. \tag{A.4}$$

These equilibrium equations are now differentiated with respect to the nodal coordinates (effectively a Jacobian matrix) which will develop into the tangent stiffness matrix,

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_{1i}}{\partial x_{1i}} & \cdots & \frac{\partial f_{1i}}{\partial x_{2k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{2k}}{\partial x_{1i}} & \cdots & \frac{\partial f_{2k}}{\partial x_{2k}} \end{bmatrix} \tag{A.5}$$

with

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

where the vectors $\mathbf{f}_a$ and $\mathbf{x}_a$ are 3-dimensional, with corresponding indices i, j, k.

For the sake of brevity, the calculation will only be continued for the top left quarter of the Jacobian; a full derivation is provided in Guest [11]. Now, using equation A.3 we obtain

$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} \leftrightarrow \frac{\partial f_{1i}}{\partial x_{1j}} = (x_{1i} - x_{2i})\frac{\partial \hat{t}}{\partial x_{1j}} + \delta_{ij}\hat{t} \tag{A.6}$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} \equiv \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases} . \tag{A.7}$$

This can be simplified by observing that

$$\frac{\partial \hat{t}}{\partial x_{1j}} = \frac{d\hat{t}}{dl}\frac{\partial l}{\partial x_{1j}} = \frac{d\hat{t}}{dl}n_j \tag{A.8}$$

by rewriting $\frac{d\hat{t}}{dl}$ as

$$\frac{d\hat{t}}{dl} = \frac{d(t/l)}{dl} = \frac{1}{l}\left(\frac{dt}{dl} - \hat{t}\right) \tag{A.9}$$

and by considering that dt/dl is equal to the axial stiffness $g = \frac{EA_0}{l_0}$ (unless a cable is at its rest length), resulting in

$$\frac{d\hat{t}}{dl} = \frac{g - \hat{t}}{l} \tag{A.10}$$

which simplifies even further when the *modified axial stiffness* $\hat{g} = g - \hat{t}$ is introduced, giving

$$\frac{d\hat{t}}{dl} = \frac{\hat{g}}{l}. \tag{A.11}$$

Returning to equation A.6 and using the results in equations A.8 and A.11 we arrive at

$$\frac{\partial f_{1i}}{\partial x_{1j}} = n_i \hat{g} n_j + \delta_{ij}\hat{t} \tag{A.12}$$

or, in vector form

$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} = \mathbf{n}\hat{g}\mathbf{n}^T + \hat{t}\mathbf{I}. \tag{A.13}$$

When the above calculations are performed for the entire Jacobian, the tangent stiffness matrix, relating small changes in nodal position to small changes in nodal forces can be written as

$$\mathbf{K}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} [\hat{g}] \begin{bmatrix} \mathbf{n}^T & -\mathbf{n}^T \end{bmatrix} + \hat{t} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}. \tag{A.14}$$

With

$$\mathbf{a}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} \quad ; \quad \mathbf{S}_s = \hat{t} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

this becomes

$$\mathbf{K}_s = \mathbf{a}_s[\hat{g}]\mathbf{a}_s^T + \mathbf{S}_s \tag{A.15}$$

where the second part is referred to as the *stress matrix*.

It should be noted that in the case of zero-free-length springs the modified axial stiffness is equal to zero, because the tension coefficient $\hat{t}$ is idential to the *constant* spring stiffness of the zero-free-length spring, $\mathbf{K}_{zfl}$. This means that only the *stress matrix* is relevant in those situations; this implies that for zero-free-length springs merely topology and not geometry is relevant.

A.3 Geometrically non-linear FEA

The derivation of the tangent stiffness matrix in this section is largely based on calculations in Crisfield [7], but with notations adapted to those in Guest [11] for ease of comparison.

A.3.1 Strain of a bar element

The initial length l_0 and deformed length l_n of the bar are written as

$$l_0^2 = (\mathbf{x}_1 - \mathbf{x}_2)^T(\mathbf{x}_1 - \mathbf{x}_2) \tag{A.16}$$

$$l_n^2 = (\mathbf{x}'_1 - \mathbf{x}'_2)^T(\mathbf{x}'_1 - \mathbf{x}'_2) \tag{A.17}$$

where $\mathbf{x}'_a$ is the set of displaced coordinates for node a, related to the initial coordinates by the nodal displacements $\mathbf{p}_a$

$$\mathbf{x}'_a = \mathbf{x}_a + \mathbf{p}_a. \tag{A.18}$$

For computational ease, equation A.17 will henceforth be written as

$$l_n^2 = \mathbf{x}'^T\mathbf{A}\mathbf{x}' \tag{A.19}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}' = \begin{bmatrix} x_{1i} \\ x_{1j} \\ x_{1k} \\ x_{2i} \\ x_{2j} \\ x_{2k} \end{bmatrix}'. \tag{A.20}$$

The engineering strain is defined as

$$\epsilon = \frac{l_n - l_0}{l_0}. \tag{A.21}$$

Now, when keeping equations A.21 and A.19 in mind, the following can be derived:

$$\frac{\partial \epsilon^T}{\partial \mathbf{p}} = \frac{1}{l_0}\frac{\partial l_n^T}{\partial \mathbf{p}} = \frac{1}{l_0}\frac{1}{l_n}\mathbf{A}\mathbf{x}' = \frac{1}{l_0 l_n}\mathbf{c}(\mathbf{x}') \tag{A.22}$$

with

$$\mathbf{c}(\mathbf{x}') = \begin{bmatrix} x'_{12i} \\ x'_{12j} \\ x'_{12k} \\ -x'_{12i} \\ -x'_{12j} \\ -x'_{12k} \end{bmatrix} \text{ and } x'_{12i} = x'_{1i} - x'_{2i}. \tag{A.23}$$

Note the similarity of $\mathbf{c}(\mathbf{x}')$ with the equilibrium vector $\mathbf{a}_s$ defined in section A.2. Their relation can be written as

$$\mathbf{c}(\mathbf{x}') = l_n \begin{bmatrix} \mathbf{n}_n \\ -\mathbf{n}_n \end{bmatrix} = l_n \mathbf{a}_s. \tag{A.24}$$

A.3.2 Equilibrium equations

The virtual work principle will be used to derive the equilibrium equations,

$$\delta W_u = \delta W_i$$

$$\delta \mathbf{p}^T \mathbf{q} = \delta \mathbf{p}^T \int \sigma \frac{\partial \epsilon}{\partial \mathbf{p}} dV$$

$$\mathbf{q} = \int \sigma \frac{\partial \epsilon}{\partial \mathbf{p}} dV = A_0 l_0 \sigma \mathbf{b} = \frac{\sigma A_0}{l_n}\mathbf{c}(\mathbf{x}') = \lambda \frac{\sigma A_0}{l_0}\mathbf{c}(\mathbf{x}') \tag{A.25}$$

with

$$\lambda = \frac{l_0}{l_n}. \tag{A.26}$$

A.3.3 Tangent stiffness matrix

Presently the tangent stiffness matrix can be derived,

$$\mathbf{K}_t = \frac{\partial \mathbf{q}}{\partial \mathbf{p}} = \frac{A_0}{l_n}\mathbf{c}(\mathbf{x}')\frac{\partial \sigma^T}{\partial \mathbf{p}} + \frac{\sigma A_0}{l_n}\frac{\partial \mathbf{c}(\mathbf{x}')}{\partial \mathbf{p}} - \frac{\sigma A_0}{l_n^2}\mathbf{c}(\mathbf{x}')\frac{\partial l_n}{\partial \mathbf{p}}. \tag{A.27}$$

It is clear that the tangent stiffness matrix consists of three individual matrices; the first is the *linear stiffness matrix*, the other two form the *geometric stiffness matrix*

$$\mathbf{K}_t = \mathbf{K}_{t1} + \mathbf{K}_{t\sigma 1} + \mathbf{K}_{t\sigma 2}. \tag{A.28}$$

Linear stiffness matrix

The first term in equation A.27 is the linear stiffness matrix, and when taking into account that

$$\frac{\partial \sigma}{\partial \mathbf{p}} = E\frac{\partial \epsilon}{\partial \mathbf{p}} = E\mathbf{b}^T = \frac{E}{l_0 l_n}\mathbf{c}(\mathbf{x}')^T \tag{A.29}$$

the linear stiffness matrix becomes

$$\mathbf{K}_{t1} = \frac{EA_0}{l_0 l_n^2}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T = \frac{EA_0}{l_0^3}\lambda^2\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T. \tag{A.30}$$

Geometric stiffness matrix

Keeping in mind that $\frac{\partial \mathbf{c}(\mathbf{x}')}{\partial \mathbf{p}} = \mathbf{A}$ the second term in equation A.27 can be rewritten as

$$\mathbf{K}_{t\sigma 1} = \frac{\sigma A_0}{l_0}\lambda\mathbf{A}. \tag{A.31}$$

The last term in equation A.27 can also be rewritten, because with equation A.22 it holds that

$$\frac{\partial l_n}{\partial \mathbf{p}} = \frac{1}{l_n}\mathbf{c}(\mathbf{x}')^T \tag{A.32}$$

which results in

$$\mathbf{K}_{t\sigma 2} = -\frac{\sigma A_0}{l_n^2}\mathbf{c}(\mathbf{x}')\frac{\partial l_n}{\partial \mathbf{p}} = -\frac{\sigma A_0}{l_0^3}\lambda^3\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T. \tag{A.33}$$

A.4 Comparison

Now it is time to compare the tangent stifness matrix found in Guest [11] to the one derived in the previous section

$$\mathbf{K}_T = \mathbf{a}_s[\hat{g}]\mathbf{a}_s^T + \hat{t}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}. \tag{A.34}$$

Here

$$\mathbf{a}_s = \begin{bmatrix} \mathbf{n} \\ -\mathbf{n} \end{bmatrix} \quad ; \quad \hat{t} = \frac{t}{l} \quad ; \quad \mathbf{S}_s = \hat{t}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

and importantly the *modified axial stiffness* is

$$\hat{g} = g - \hat{t} = \frac{EA_0}{l_0} - \hat{t}. \tag{A.35}$$

These results are now to be compared with equations A.30, A.31 and A.33

$$\mathbf{K}_{t1} = \frac{EA_0}{l_0^3}\lambda^2\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T$$

$$\mathbf{K}_{t\sigma1} = \frac{\sigma A_0}{l_0}\lambda\mathbf{A}$$

$$\mathbf{K}_{t\sigma2} = -\frac{\sigma A_0}{l_0^3}\lambda^3\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T.$$

Bearing in mind equation A.26, $\lambda = \frac{l_0}{l_n}$, the equations become

$$\mathbf{K}_{t1} = \frac{EA_0}{l_0 l_n^2}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T \tag{A.36}$$

$$\mathbf{K}_{t\sigma1} = \frac{\sigma A_0}{l_n}\mathbf{A} \tag{A.37}$$

$$\mathbf{K}_{t\sigma2} = -\frac{\sigma A_0}{l_n^3}\mathbf{c}(\mathbf{x}')\mathbf{c}(\mathbf{x}')^T. \tag{A.38}$$

When using equation A.24, $\mathbf{c}(\mathbf{x}') = l_n\mathbf{a}_s$, these can be rewritten into

$$\mathbf{K}_{t1} = \frac{EA_0}{l_0}\mathbf{a}_s\mathbf{a}_s^T \tag{A.39}$$

$$\mathbf{K}_{t\sigma 1} = \frac{\sigma A_0}{l_n}\mathbf{A} \tag{A.40}$$

$$\mathbf{K}_{t\sigma 2} = -\frac{\sigma A_0}{l_n}\mathbf{a}_s\mathbf{a}_s^T. \tag{A.41}$$

Finally, when it is observed that $\frac{\sigma A_0}{l_n}$ is equal to $\hat{t}$ and using $g = \frac{EA_0}{l_0}$ the equations can be rewritten as:

$$\mathbf{K}_{t1} = \mathbf{a}_s[g]\mathbf{a}_s^T \tag{A.42}$$

$$\mathbf{K}_{t\sigma 1} = \hat{t}\mathbf{A} \tag{A.43}$$

$$\mathbf{K}_{t\sigma 2} = -\mathbf{a}_s[\hat{t}]\mathbf{a}_s^T. \tag{A.44}$$

Now it is obvious that $\mathbf{K}_{t\sigma 1}$ is identical to the stress matrix $\mathbf{S}_s$ and when equations A.42 and A.44 are combined they produce the first term from equation A.34.

A.5 Conclusion

So concluding, the two approaches to the tangent stiffness matrix provide identical results, given the assumption that the length l in Guest [11] is equal to l_n. This is the case as the bar is assumed to be pretensioned and is thus already in a deformed state.

By introducing the *modified axial stiffness*, the derivation used in Guest [11] provides a valuable insight regarding the use of zero-free-length springs, which is harder to recognize when using the traditional non-linear FEM approach to the tangent stiffness matrix.

It must be added that the use of a global reference frame makes it easier to arrive at the tangent stiffness formulation mentioned above, compared to the use of local coordinates on the bars and performing coordinate transformations at a later stage, section 3.6 in Crisfield [7].

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